

Two-mode spin-squeezing as a resource for multiparameter quantum estimation

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We study the estimation of a two-phase rotation in a finite dimensional spin system. We consider two-mode spin squeezed probe states and discuss the role of two-mode spin squeezing in this kind of estimation. After deriving the ultimate bounds posed by quantum mechanics for such probe states, we present a measurement scheme able to exploit two-mode spin squeezing in order to improve the measurement sensitivity. We discuss the case where, even in the presence of two-mode spin squeezing, the coherent state strategy yields a better performance and thus the standard quantum limit is not overcome. Then, by considering optimized two-mode squeezed probe states, we show that our measurement scheme always beats the standard quantum limit, and approaches the ultimate bounds for large values of the spin dimension.

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I. INTRODUCTION

Continuous variable (CV) squeezing [1, 2], also known as *bosonic* squeezing, has played a major role in the development of modern quantum optics, and it is now safe to say that a good level of understanding has been reached regarding its fundamental and practical importance [3]. Its study has been somewhat facilitated by the fact that the uncertainty relation between two canonically conjugate operators is state-independent, hence a universal notion of squeezing arises naturally and its quantification via a ‘squeezing parameter’ is essentially unique (up to monotone re-scalings).

More recently, increasing attention has been devoted to the finite-dimensional counterpart of CV squeezing, that is, *Spin Squeezing* [4, 5]. This has been mainly motivated by the study of correlations and entanglement between particles [7, 8], and by the quest to increase the precision of measurements in experiments involving finite-dimensional systems [3, 9]. In the study of spin squeezing, the first noticeable difference as compared to the CV case is that the uncertainty relation between two angular momentum operators is state dependent, hence several inequivalent squeezing parameters can be defined [6]. Typically, the usefulness and relevance of each one of these quantifiers depends on the considered application, and the performance of different tasks will be related to different squeezing parameters. To mention some notable examples, spin-squeezing and its various quantifiers have proven useful tools for the detection of quantum entanglement in many-body systems [7, 8], and for increasing the precision of instruments such as Ramsey spectrometers [5, 9, 10], atomic clocks [5, 8] and ultra-sensitive magnetometers [11]. Even applications as broad as quantum computation and quantum simulation have been shown to benefit from the presence of spin squeezing [12], and the latter is becoming progressively easier to be generated and measured experimentally: in particular in atomic ensembles [13], Bose-Einstein condensates [14] and atomic chips [15].

Considering bipartite systems, one can also introduce the con-

cept of two-mode spin squeezing (TMSS), as the generalization of its bosonic counterpart. Two mode spin squeezed states exhibit squeezing in their collective spin degrees of freedom, but not in the local ones. TMSS has been widely investigated [16–20] and for pure states it has been proven to be equivalent (up to local unitaries) to bipartite entanglement for systems with equal spin [21].

Here, we are particularly interested in the application of spin squeezing to the rapidly developing field of *quantum metrology*, which studies how the peculiar features of quantum mechanics affect the achievable precision in the estimation of one or more parameters. In doing so, we shall use the theoretical tools provided by *quantum estimation theory* [22, 23], which gives fundamental bounds to the achievable accuracies in terms of the Quantum Fisher Information (QFI).

In the context of quantum metrology with bosonic systems, squeezing is well established as being a desirable feature, able to enhance the performance of single-parameter estimation protocols [24–28]. On the other hand two-mode squeezed states have been proven to be useful resources for several estimation problems such as the joint estimation of loss and temperature of a bosonic channel [29, 30] or the characterization of both parameters defining a displacement operation [31, 32]. Similarly, spin squeezing has been identified as a crucial resource in several single-parameter estimation protocols, which typically boil down to determining the angle of rotation of a spin around some known axis [5].

In this paper, we contribute to these developments by investigating the resource power of TMSS in a multi-parameter quantum estimation problem. The estimation of multiple parameters is receiving increasing attention in the literature (see for example [22, 29, 30, 32, 34, 35, 37–42]). This kind of estimation is fundamentally different from the single-parameter case, as different parameters may be associated to mutually non-commuting operators. Hence, the theoretical bounds provided by local estimation theory may not be achievable, as they may assume measurement strategies that go beyond the operations allowed by quantum mechanics.

More specifically, we study a broad class of two-mode spin squeezed states. We show how these states allow the efficient estimation of two unknown phases characterizing a spin rotation, by making use of a simple strategy in which the rota-

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tion is applied to only one of the two subsystems, followed by a projective measurement on the local spin operators of both parties. By accurate numerical calculations we show that two-mode spin squeezing is indeed a key resource in the considered problem, allowing to beat the standard quantum limit (SQL) which, for spin systems, is obtained by considering the estimation power of coherent spin states (CSS). Moreover, we show how our considered strategy becomes nearly-optimal when the dimensionality of the spins becomes large. The paper is organized as follows: in the next section we introduce the concept of spin squeezing, giving its definition in terms of a metrological application, along with the definition of two-mode spin squeezing. In Sec. III we provide a brief introduction to local quantum estimation theory, both for the single- and the multi-parameter case. In Sec. IV we discuss in detail the estimation problem at the center of our investigations. We propose a measurement strategy for the estimation of the relevant parameters, and we compare our results to the ultimate limits posed by quantum mechanics. Sec. V concludes the paper with some remarks.

II. SPIN SQUEEZING

Let us consider a spin- j system, characterized by the angular momentum operators $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$, which satisfy the commutation rule

$$[\hat{J}_\alpha, \hat{J}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{J}_\gamma. \quad (1)$$

where α, β, γ denote the components in any three orthogonal directions, and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. This implies an uncertainty relation

$$\Delta\hat{J}_x\Delta\hat{J}_y \geq \frac{|\langle\hat{J}_z\rangle|}{2}, \quad (2)$$

where $\Delta\hat{J}_\alpha^2 = \langle\hat{J}_\alpha^2\rangle - \langle\hat{J}_\alpha\rangle^2$ and $\langle O \rangle = \text{Tr}[\rho O]$ denotes the expectation value of an operator on a given quantum state ρ . To better understand spin squeezing, it is useful to first introduce the notion of coherent spin states (CSS). These are defined as

$$|C(\theta, \phi)\rangle = \hat{R}(\theta, \phi)|j, j\rangle \quad (3)$$

where $|j, j\rangle$ is the eigenstate of \hat{J}_z with eigenvalue j and

$$\hat{R}(\theta, \phi) = \exp(-i\phi\hat{J}_n) = \exp[i\phi(\hat{J}_x \sin \theta - \hat{J}_y \cos \theta)] \quad (4)$$

represents a rotation of angle ϕ around the axis $\vec{n} = (-\sin \theta, \cos \theta, 0)^\top$. Different definitions of spin squeezing have been introduced, the most used being the ones suggested by Kitagawa and Ueda [4], and by Wineland *et al.* [5] (see [6] for a complete review of these definitions). In the following we will present the details regarding the latter definition [5], which is more relevant for metrological applications. Let us start by defining the mean-spin direction (MSD)

$$\vec{n} = \frac{\langle\hat{\vec{J}}\rangle}{|\langle\hat{\vec{J}}\rangle|} \quad (5)$$

where $\hat{\vec{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)^\top$. Without loss of generality, let us consider a state with MSD along the z -axis. This state is then rotated by a phase ϕ around the x -axis, as described by the unitary operator $\hat{R}(-\pi/2, \phi)$. For small values of the angle ϕ , one can show that the phase sensitivity, obtained by measuring the output spin operator $\hat{J}_y^{\text{out}} = \hat{R}(-\pi/2, \phi)^\dagger \hat{J}_y \hat{R}(-\pi/2, \phi)$, reduces to

$$\Delta\phi = \frac{\Delta\hat{J}_y^{\text{out}}}{|\partial_\phi \langle\hat{J}_y^{\text{out}}\rangle|} \approx \frac{\Delta\hat{J}_y}{|\langle\hat{J}_z\rangle|}, \quad (6)$$

and is thus related to the initial variance of the operator \hat{J}_y . For a coherent spin state, the fluctuations of all the spin operators orthogonal to the MSD are equal to $(\Delta\hat{J}_{\vec{n}_\perp})^2 = j/2$ and then one proves that

$$\Delta\phi_{\text{SQL}} = \frac{1}{\sqrt{2j}}, \quad (7)$$

which is referred to as the standard quantum limit (SQL) or shot-noise limit. In analogy to the CV case, a quantum state is spin-squeezed if it exhibits lower fluctuations in one of the directions orthogonal to the MSD; according to this definition, a state is said to be spin squeezed if and only if its corresponding phase sensitivity is below the shot noise limit, *i.e.*

$$\Delta\phi \leq \Delta\phi_{\text{SQL}}. \quad (8)$$

One can show that this can be obtained if the fluctuations of the spin operator \hat{J}_y are below a certain threshold, *i.e.* $\Delta\hat{J}_y^2 < j/2$.

Combining the uncertainty relation (2) with the inequality $j^2 \geq \langle\hat{J}_\alpha^2\rangle \geq \Delta\hat{J}_\alpha^2$, which bounds the maximum variance on a spin observable, one also obtains the *Heisenberg limit* (HL)

$$\Delta\phi \geq \Delta\phi_{\text{HL}} = \frac{1}{2j}, \quad (9)$$

posing the ultimate limit on the estimation precision of the phase ϕ compatible with quantum mechanics.

Let us now move on to the definition of two-mode spin squeezing [16–20] for a bipartite spin system described by operators $\hat{\vec{J}}_a = (\hat{J}_{x_a}, \hat{J}_{y_a}, \hat{J}_{z_a})^\top$ with $a = \{1, 2\}$. Sums and differences of spin operators belonging to different systems are denoted by $\hat{J}_{\alpha\pm} = \hat{J}_{\alpha_1} \pm \hat{J}_{\alpha_2}$, and obey the uncertainty relations

$$\Delta\hat{J}_{x\pm}\Delta\hat{J}_{y\pm} \geq \frac{|\langle\hat{J}_{z+}\rangle|}{2}. \quad (10)$$

A state is said to be two-mode spin squeezed if

$$(\Delta\hat{J}_{x-})^2 + (\Delta\hat{J}_{y+})^2 \leq |\langle\hat{J}_{z+}\rangle|. \quad (11)$$

Here, the reduced fluctuations are observed in the non-local operators \hat{J}_{x-} and \hat{J}_{y+} . In fact, two-mode spin squeezing is not only a sufficient [19, 20], but also a necessary condition for entanglement in pure states of two subsystems with equal spin (except for a set of bipartite states with measure zero) [21].

III. QUANTUM ESTIMATION THEORY

We will give here a brief review of quantum estimation theory [22, 23, 43]. Let us consider a family of quantum states ϱ_λ , labelled by a real parameter λ that we aim to estimate. The ultimate limit to the estimation precision of the parameter λ is given by the quantum Cram r-Rao bound

$$\text{Var}(\lambda) \geq \frac{1}{MH(\lambda)}, \quad (12)$$

where M is the number of measurements performed,

$$H(\lambda) = \text{Tr}[\varrho_\lambda L_\lambda^2] \quad (13)$$

is the quantum Fisher information and L_λ is the Symmetric Logarithmic Derivative (SLD), that is, the operator satisfying

$$2\partial_\lambda \varrho_\lambda = L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda. \quad (14)$$

Note that, in principle, one can always find a quantum measurement able to attain equality in Eq. (12), hence saturating the quantum Cram r-Rao bound.

Moving on to a multiparameter scenario, let us consider a family of quantum states $\varrho_{\mathbf{z}}$ labelled by d different parameters $\mathbf{z} = \{z_\mu\}$, $\mu = 1, \dots, d$. The SLD for each parameter is defined via

$$\frac{\partial \varrho_{\mathbf{z}}}{\partial z_\mu} = \frac{L_\mu^{(S)} \varrho_{\mathbf{z}} + \varrho_{\mathbf{z}} L_\mu^{(S)}}{2}, \quad (15)$$

from which one can calculate the QFI matrix \mathbf{H} :

$$\mathbf{H}_{\mu\nu} = \text{Tr} \left[\varrho_{\mathbf{z}} \frac{L_\mu^{(S)} L_\nu^{(S)} + L_\nu^{(S)} L_\mu^{(S)}}{2} \right]. \quad (16)$$

We define the covariance matrix elements $V(\mathbf{z})_{\mu\nu} = E[z_\mu z_\nu] - E[z_\mu]E[z_\nu]$ and consider a weight (positive definite) matrix \mathbf{G} . Then, the multiparameter quantum Cram r-Rao bounds read

$$\text{tr}[\mathbf{G}\mathbf{V}] \geq \frac{1}{M} \text{tr}[\mathbf{G}(\mathbf{H})^{-1}], \quad (17)$$

where $\text{tr}[A]$ is the trace operation on a finite dimensional matrix A and M is the number of measurements performed. We observe that if we choose $\mathbf{G} = \mathbb{1}$ we obtain the bound on the sum of the variances of the parameters involved,

$$\sum_\mu \text{Var}(z_\mu) := \frac{(\Delta \mathbf{z})^2}{M} \geq \frac{1}{M} \text{tr}[\mathbf{H}^{-1}] \quad (18)$$

where we have introduced the *overall* multiparameter sensitivity $\Delta \mathbf{z}$. Differently from the single parameter case, the multiparameter bound is not always achievable, since optimal measurements for different parameters may correspond to non-commuting observables. We remark that other (not always achievable) bounds to the estimation precision have been introduced in the multiparameter case, together with the concept of *most informative* bound [34–38]. In this manuscript we shall focus on the standard Cram r-rao bound described above.

IV. TWO-PHASES QUANTUM ESTIMATION

We consider the following estimation problem: a probe state ϱ undergoes a unitary evolution

$$\hat{U}(\phi_x, \phi_y) = \exp\{i(\phi_x \hat{J}_y + \phi_y \hat{J}_x)\}, \quad (19)$$

characterized by two different phases ϕ_x and ϕ_y which we aim to estimate. In the case of a physical spin, such rotation could be the result of the interaction between the system and a classical magnetic field lying in the XY plane. We shall work in the Heisenberg picture, characterizing the action of the operator in Eq. (19) via

$$\hat{J}^{\text{out}} = \hat{U}(\phi_x, \phi_y)^\dagger \hat{J} \hat{U}(\phi_x, \phi_y) = M(\phi_x, \phi_y) \hat{J} \quad (20)$$

where $\hat{J} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)^\top$. $M(\phi_x, \phi_y)$ is the 3×3 rotation matrix associated to the operator $\hat{U}(\phi_x, \phi_y)$: its expression is formally equivalent to Eq. (19), provided that \hat{J}_x and \hat{J}_y are replaced by the corresponding 3×3 generators of the rotation group. We are interested in a regime where the values of the parameters ϕ_x and ϕ_y are small: in this case the matrix $M(\phi_x, \phi_y)$ can be expanded at first order in the two phases, yielding

$$M(\phi_x, \phi_y) \approx \begin{pmatrix} 1 & 0 & \phi_x \\ 0 & 1 & -\phi_y \\ -\phi_x & \phi_y & 1 \end{pmatrix}. \quad (21)$$

As a consequence one has

$$\hat{J}_x^{\text{out}} \approx \hat{J}_x + \phi_x \hat{J}_z \quad (22)$$

$$\hat{J}_y^{\text{out}} \approx \hat{J}_y - \phi_y \hat{J}_z, \quad (23)$$

showing that by measuring the angular momentum operator \hat{J}_x and \hat{J}_y after the rotation one gains information about the corresponding phases.

Following the results obtained for multi-parameter estimation by using two-mode squeezed entangled states in continuous-variable systems [29, 30, 32], we will consider the case where the probe state ϱ is a bipartite system and each subsystem α is described by the angular momentum operators vector $\hat{J}_\alpha = (\hat{J}_{x_\alpha}, \hat{J}_{y_\alpha}, \hat{J}_{z_\alpha})^\top$. The estimation strategy is described in Fig. 1: a bipartite state is prepared and the rotation operator $\hat{U}(\phi_x, \phi_y)$ is applied on one of the two arms. After the rotation, spin measurements are performed on the two subsystems. Before specifying the measurements that have to be performed, let us focus on the probe state. We look for bipartite states which exhibit two-mode spin-squeezing, that is, they satisfy Eq. (11), and in order to do this we follow the approach of Ref. [19]. We thus consider states in the Schmidt form

$$|\Psi_{\text{in}}\rangle = \sum_{m=-j}^{m=j} \Phi_m |j, m\rangle |j, m\rangle. \quad (24)$$

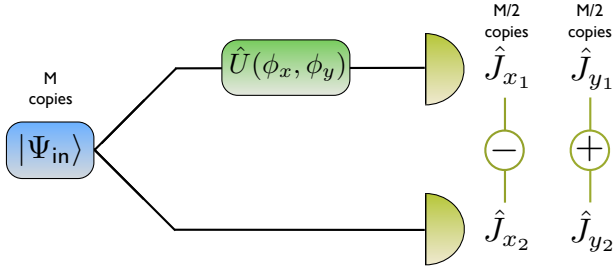


Figure 1: Measurement strategy for a two-phases measurement. A two-mode spin squeezed state is used as a probe. On one arm the two-phases rotation is performed; after the rotation measurements of alternatively \hat{J}_{x1} and \hat{J}_{x2} or \hat{J}_{y1} and \hat{J}_{y2} are performed on the two output systems. The results are then combined to obtain \hat{J}_{x-} and \hat{J}_{y+} .

From Eq. (24), one can immediately deduce the following relations between the expectation values of local spin observables:

$$\langle \hat{J}_{z1} \rangle = \langle \hat{J}_{z2} \rangle, \quad (25)$$

$$\langle \hat{J}_{x1} \rangle = \langle \hat{J}_{x2} \rangle = \langle \hat{J}_{y1} \rangle = \langle \hat{J}_{y2} \rangle = 0. \quad (26)$$

Within the class of states defined by Eq. (24), we can identify two-mode spin squeezed states by fixing a real parameter μ , solving the eigenvalue problem

$$(\hat{J}_{x-}^2 + \hat{J}_{y+}^2 - \mu \hat{J}_{z+}) |\Psi_{\text{in}}\rangle = \nu |\Psi_{\text{in}}\rangle, \quad (27)$$

and choosing the eigenvector corresponding to the minimum eigenvalue ν_{min} . When this procedure is repeated for different values of μ , it is found that the corresponding output state yields different values of $\langle \hat{J}_{z+} \rangle = 2\langle \hat{J}_{z1} \rangle$. Hence, the expectation value $\langle \hat{J}_{z1} \rangle$ can be used to label the two-mode spin-squeezed states produced via such algorithm [19]. If at least two coefficients Φ_m are not null, it is known that the state will exhibit two-mode spin squeezing [21]. Applying Eq. (20) to the first subsystem, together with the small angles approximation [Eq. (21)], we obtain

$$\langle \hat{J}_{x-}^{\text{out}} \rangle \approx \phi_x \langle \hat{J}_{z1} \rangle \quad \langle \hat{J}_{y+}^{\text{out}} \rangle \approx -\phi_y \langle \hat{J}_{z1} \rangle. \quad (28)$$

As a consequence, we can obtain information about the phase values from the measurement of the two observables J_{x-} and J_{y+} . Moreover, since the variances of these observables are not changed by such rotation, *i.e.*

$$(\Delta \hat{J}_{x-}^{\text{out}})^2 \approx (\Delta \hat{J}_{x-})^2, \quad (29)$$

$$(\Delta \hat{J}_{y+}^{\text{out}})^2 \approx (\Delta \hat{J}_{y+})^2, \quad (30)$$

one can use the TMSS property (11) to perform a precise estimation of the two parameters. In particular, the intrinsic phase sensitivities of the probe state read

$$\Delta \phi_x = \frac{\Delta \hat{J}_{x-}}{|\langle \hat{J}_{z1} \rangle|}, \quad (31)$$

$$\Delta \phi_y = \frac{\Delta \hat{J}_{y+}}{|\langle \hat{J}_{z1} \rangle|}. \quad (32)$$

The estimation strategy in Fig. 1 resembles to the one used in [32] for the two-parameter displacement estimation with continuous-variable systems. However, in that case one is able to associate the two parameters to two commuting observables (at the cost of adding some extra noise), which can then be measured simultaneously. Here, the two observables $\hat{J}_{x-}^{\text{out}}$ and $\hat{J}_{y+}^{\text{out}}$ commute only on average, and in such a case the question of simultaneous measurement becomes nontrivial [33]. We then choose a measurement strategy which is motivated by simplicity: given M copies of the probe state, on $M/2$ of those we perform the measurement of $\hat{J}_{x1,2}$, while the operators $\hat{J}_{y1,2}$ are measured on the remaining $M/2$ copies. Then, $M/2$ values for \hat{J}_{x-} and \hat{J}_{y+} are obtained respectively by subtracting or summing the experimental outcomes. When $M \gg 1$, the Central Limit Theorem predicts the total experimental variance of the estimated parameters, as per

$$\text{Var}(\phi_x) + \text{Var}(\phi_y) = \frac{2}{M} (\Delta \phi_x^2 + \Delta \phi_y^2). \quad (33)$$

A comparison of this expression with the Cramér-Rao bound in Eq. (18), which is also expressed in terms of the *total* number of measurements M , leads us naturally to define the *effective* phase sensitivities

$$\Delta \phi_x^{\text{eff}} = \sqrt{2} \Delta \phi_x \quad \Delta \phi_y^{\text{eff}} = \sqrt{2} \Delta \phi_y. \quad (34)$$

Then, the performance of our estimation protocol can be quantified via the two-phase sensitivity

$$\begin{aligned} \Delta \Phi &= \sqrt{(\Delta \phi_x^{\text{eff}})^2 + (\Delta \phi_y^{\text{eff}})^2} \\ &= \sqrt{2 \left(\frac{(\Delta \hat{J}_{x-})^2}{|\langle \hat{J}_{z1} \rangle|^2} + \frac{(\Delta \hat{J}_{y+})^2}{|\langle \hat{J}_{z1} \rangle|^2} \right)} \end{aligned} \quad (35)$$

Our aim is now to compare the phase sensitivity in Eq. (35) to the ultimate bounds imposed by quantum estimation theory. The HL, representing the best sensitivity compatible with quantum mechanics, can be easily calculated by noting that the uncertainty relations in Eq. (10), together with Eq. (25) and the condition $\Delta \hat{J}_{\alpha\pm} \leq 2j$, imply $\Delta \hat{J}_{x-} \geq |\langle \hat{J}_{z1} \rangle|/2j$ and also $\Delta \hat{J}_{y+} \geq |\langle \hat{J}_{z1} \rangle|/2j$, so that the minimum value that Eq. (35) can take compatibly with these constraints is

$$\Delta \Phi_{\text{HL}} = \frac{1}{j}. \quad (36)$$

The SQL represents the estimation sensitivity which is achievable with ‘classical’ resources, that is, coherent spin states. One may be tempted to calculate such quantity by evaluating Eq. (35) for a CSS such as $|j, j\rangle \otimes |j, j\rangle$. This, however, would underestimate the resource power of CSSs: we recall that in our protocol the spin rotation (19) is applied only to the first subsystem, so that the measurement of the second subsystem cannot yield information about such rotation when the resource state is a two mode CSS, and it just results in extra noise being added to the experimental data. Then, to obtain the correct SQL one has to simply neglect the second

subsystem, and consider the estimation problem of a spin- j CSS to which the rotation (19) is applied, followed by the measurement of either \hat{J}_x or \hat{J}_y (each observable being measured on half of the copies). Following the same steps that lead to the single-parameter SQL in Eq. (7), one derives the two-parameter SQL as per

$$\Delta\Phi_{\text{SQL}} = \frac{\sqrt{2}}{\sqrt{j}}. \quad (37)$$

Finally, one can evaluate the Cramèr-Rao bound for the class of probe states described by Eq. (24). The QFI matrix for the two-phases estimation problem reads

$$\mathbf{H} = \begin{pmatrix} \langle \hat{J}_{x_1}^2 \rangle & \langle \hat{J}_{x_1} \hat{J}_{y_1} + \hat{J}_{y_1} \hat{J}_{x_1} \rangle / 2 \\ \langle \hat{J}_{x_1} \hat{J}_{y_1} + \hat{J}_{y_1} \hat{J}_{x_1} \rangle / 2 & \langle \hat{J}_{y_1}^2 \rangle \end{pmatrix}, \quad (38)$$

where we have used Eq. (26). As a consequence, one obtains

$$(\Delta\Phi)^2 \geq \text{Tr}[\mathbf{H}^{-1}] = \frac{1}{j^2 + j - \langle \hat{J}_{z_1}^2 \rangle}. \quad (39)$$

The Cramèr-Rao bound in Eq. (39) would suggest that the state yielding the best two-phase sensitivity should have $\langle \hat{J}_{z_1}^2 \rangle = 0$, thus implying $\langle \hat{J}_{z_1} \rangle = 0$. However, by applying our measurement strategy to such a state, it is easy to see that no information can be gained about the two phases ϕ_x, ϕ_y [see Eq. (28)], so that one obtains a divergent phase sensitivity. This clearly implies that such limit cannot be reached. Hence, to find the optimal probe state for a given spin number j we can directly study the phase sensitivity $\Delta\Phi$ in Eq. (35), as a function of the parameter $\langle \hat{J}_{z_1} \rangle$ (we recall that such mean value uniquely labels our probe states). The typical behaviour of the phase sensitivity as a function of $\langle \hat{J}_{z_1} \rangle$ is shown in Fig. 2. We observe that even if the choice $\langle \hat{J}_{z_1} \rangle = 0$ would lead to a diverging $\Delta\Phi$, the optimal probe state corresponds to a value of $\langle \hat{J}_{z_1} \rangle$ which is close to zero, as it is somehow suggested by the Cramèr-Rao bound in Eq. (39).

This behaviour is confirmed by observing Fig. 3, where the phase sensitivity is plotted against the entanglement of the probe state, quantified via the normalized von Neumann entropy of the reduced state

$$E[\Psi_{\text{in}}] = \frac{S(\text{Tr}_B[|\Psi_{\text{in}}\rangle\langle\Psi_{\text{in}}|])}{\log_2(2j+1)}, \quad (40)$$

with $S(\varrho) = -\text{Tr}[\varrho \log_2 \varrho]$. (41)

Given the mentioned equivalence between two-mode spin squeezing and entanglement [21], and the pivotal role of the latter in quantum technologies, one may suspect that entanglement itself might be the essential resource in the considered protocol. However, the examination of Fig. 3 reveals that this would be an oversimplification: even though the minimum of $\Delta\Phi$ is found in the region of high entanglement, the phase sensitivity rapidly diverges in correspondence with the maximum entanglement achievable, which in fact corresponds to the ‘phase insensitive’ state $\langle \hat{J}_{z_1} \rangle = 0$ discussed above. This

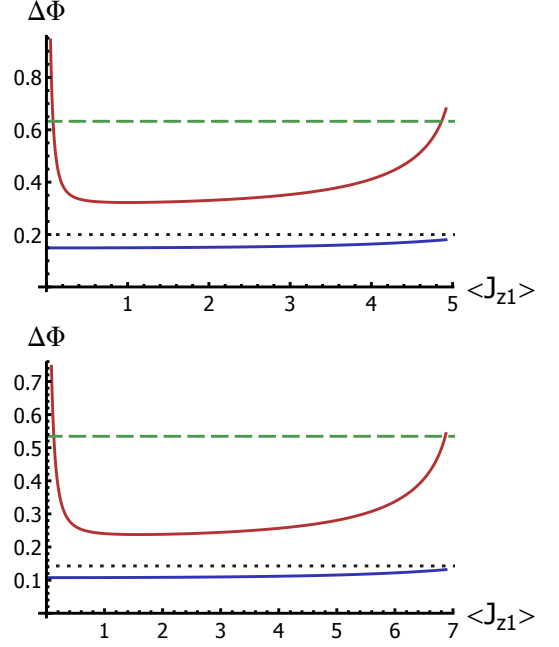


Figure 2: (Red upper solid line) Phase sensitivity $\Delta\Phi$ and (blue lower solid line) Cramèr-Rao bound for the state in Eq. 24, as a function of $\langle \hat{J}_{z_1} \rangle$, along with the SQL (dashed green line) and the HL (dotted black line), for different values of the spin dimension j . Top: $j = 5$; bottom: $j = 7$.

clearly shows that bipartite entanglement, even if it can be exploited to the enhancement in the estimation precision, is not the sole resource for this particular task, an important role being played by the detailed structure of the considered states.

It can be observed that the considered two-mode squeezed states, combined with our measurement strategy, allow us to beat the SQL in a wide parameter range, with the only exception of the two regions $\langle \hat{J}_{z_1} \rangle \sim 0$ and $\langle \hat{J}_{z_1} \rangle \lesssim j$. Note in particular that when $\langle \hat{J}_{z_1} \rangle \rightarrow j$ the probe state converges to a CSS, while the phase sensitivity *does not* converge to the SQL: our measurement strategy is in fact sub-optimal when applied to CSSs (see derivation of the SQL above), and it becomes advantageous only when the TMSS property compensates for the extra noise introduced by measuring the second subsystem. In Fig. 4, we plot the optimized phase sensitivity as a function of the spin number j , comparing it to the Cramèr-Rao bound, the Heisenberg limit and the Standard quantum limit. We see that our estimation protocol yields a phase sensitivity which is well below the standard quantum limit. We also note that our strategy appears to be closer to optimality as the spin number j is increased. Indeed, while for small values of j there is a visible gap between our phase sensitivity and both the Heisenberg limit and the Cramèr-Rao bound, this difference significantly decreases upon increasing the dimension of the Hilbert space. We recall that for multi-parameter estimation the Cramèr-Rao bound obtained in Eq. (39) is not always achievable (see Sec. III). This is evident for low values of j , where the Cramèr-Rao bound is below the Heisen-

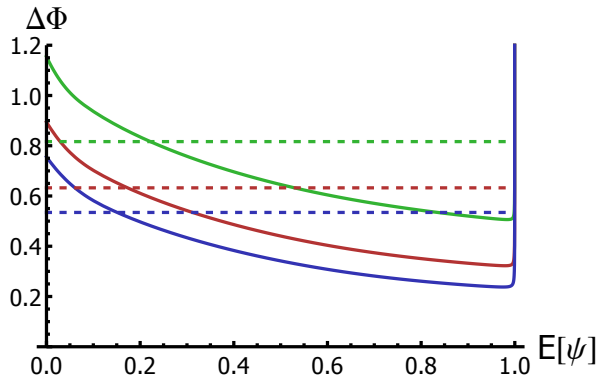


Figure 3: Phase sensitivity $\Delta\Phi$ (solid lines) as a function of the normalized entanglement $E[\Psi_{in}]$, along with the SQL (dashed lines) for different values of the spin dimension j . From top to bottom: $j = 3$ (green); $j = 5$ (red); $j = 7$ (blue).

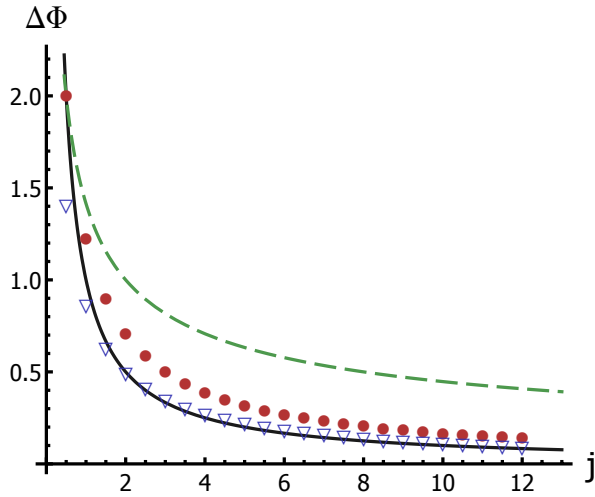


Figure 4: Phase sensitivity $\Delta\Phi$ (Red circles) and corresponding Cramèr-Rao bound (blue triangles) of optimized probe states as a function of the spin dimension j , along with Heisenberg limit (black solid line), standard quantum limit (green dashed line).

berg limit. Similarly, the two-parameter Heisenberg limit in Eq. (36) is not always achievable in principle (the optimal states for different phases may be different and the two optimal measurements may not commute). In view of the above analysis, the measurement strategy we have proposed can be considered nearly optimal for the studied estimation problem, when the spin number j is large enough.

V. CONCLUSIONS

We have studied a multiparameter quantum estimation problem in which a two-phase spin rotation is characterized by using two-mode spin states as probes. We have presented a measurement strategy which, interestingly, is sub-optimal when the resource states are classical, while at the same time is able to exploit the presence of two-mode spin squeezing to overcome the standard quantum limit. By optimizing the probe states for each spin number j , we have shown how our estimation strategy can approach the precision limits imposed by quantum estimation theory, as the spin number is increased. Our findings can be of general interest to the field of multiparameter quantum metrology with spin-squeezed resources.

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